Lecture 2  Entropy

1. The two-box model: coin flip, ideal gas in boxes, light absorption, ...

   DoF: \( n_A, n_B \)
   
   Constraints: \( N = n_A + n_B \)
   
   Extremum: \( N = N! / n_A! n_B! \).
   
   Solving \( \frac{dN}{dn_A} = 0, \frac{dN}{dn_B} = 0 \), subject to \( N = n_A + n_B \).
   
   Equilibrium state: \( n_A = n_B = N/2 \).

2. \( N \) multiplies when having two independent systems.

   Searching for an "intensive" variable: one that scales with the number of particles, one that adds with two independent systems.

   Define Entropy
   
   \[ S = k \log N \]
   
   with natural log, Boltzmann const. \( k = 1.38 \times 10^{-23} \text{ J/K} \)

   \[ \log_{10} n_A \times n_B = S_{AB} = S_A + S_B \]


   One particle having the probability of \( p_i \) in the \( i \)th box.

   Each microstate corresponds to the particle in each box, with different probabilities.

   Macrostate: \( \{ p_i \} \)

   Calculating the entropy:

   Considering \( N \) incidences of the one-particle system (ensemble), equivalent to \( N \) particles.

   \[ \{ p_i \} \xrightarrow{x_N} \{ n_i \} \]

   where \( p_i = n_i / N \) according to the definition of probability, and \( N = \sum_i n_i \)

\[ N\text{-particle multiplicity} \]

\[ n_N = N! / (n_1! n_2! ... n_i! ...) = N! / \prod_i n_i! \]

\[ N\text{-particle entropy} \]

\[ S_N = k \ln n_N = N \cdot S \]  (one particle entropy)
One-particle entropy

\[ S = \frac{1}{N} \sum \frac{N!}{\prod N \times i, N} = \frac{k}{N} \left( \ln N! - \sum \ln N \times i! \right) \]

Stirling's approximation:

\[ \frac{k}{N} \left[ N \ln N - N - \sum \ln N \times i! \right] \]

\[ N = \sum N \ln N - \sum N \times i - \sum N \ln N + \sum N \times i \]

\[ = \frac{k}{N} \sum N \times \ln N - \ln N \times i \]

\[ = k \sum \frac{N \times i}{N} \ln \frac{N \times i}{N} \]

\[ = -k \sum \frac{N \times i}{N} \ln \frac{N \times i}{N} \]

4. Entropy is defined over any probability distributions \( \{ p_i \} \)

Two-state model:

\[ \{ p_1, p_2 \} = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} \]

\[ S = 0 \quad 0.69 k \quad 0.67 k \]

Four-state model:

\[ \{ p_1, p_2, p_3, p_4 \} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \]

\[ S = 0 \quad 1.39 k \quad 0.69 k \]

5. What is entropy?

Thermodynamics: high entropy \( \rightarrow \) more disorder

Statistics: for any probability distribution \( \{ p_i \} \)

\[ S = -\sum p_i \ln p_i \]

maximize \( S \) under certain constraints gives the most probable distribution

Information theory: for a random code

\[ H = -\sum p_i \ln p_i \] (in units of bits)

\( H \) describes the information content of a message (higher \( \rightarrow \) more information)

Typical entropy of English language: \(~ 1 \text{ bit per letter. (highly redundant)}\)